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DYNAMIC MODELLING OF A NON-CONSERVATIVE DISCRETE-CONTINUOUS SYSTEM†

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Using a discrete-continuous model consisting of a viscoelastic rod with an absolutely rigid body on the end of it loaded with a follower force as the example, dynamic modelling of the stability and the transient impulse functions of the system is carried out based on the solution of the equations of motion.

The effect of dissipative forces and concentrated masses on the stability of non-conservative continuous models with a finite-dimensional approximation with respect to the first natural modes was investigated earlier in [1–3].

1. EQUATIONS OF MOTION

Suppose a homogeneous, rectilinear, viscoelastic rod of length l, with Voigt internal friction, is clamped to a fixed base in a cantilever manner (Fig. 1) and loaded with a follower force P. An absolutely rigid body of mass M and moment of inertia A is then fixed at its centre of mass onto the end of the rod. The equations of motion of the discrete-continuous system (DCS) being considered under the action of a small force $F(T)$ (T is the time), in dimensionless variables and parameters and linearized in the neighbourhood of the zero state $Y(Z, T) = \Phi(T) = 0$, have the form

$$m\frac{d^2y_1(t)}{dt^2} = n(t)+f(t), \quad a\frac{d^2\varphi(t)}{dt^2} = b(t)$$

$$\left(1+\gamma\frac{\partial}{\partial t}\right)\frac{\partial^4 y(z,t)}{\partial z^4} + p\frac{\partial^2 y(z,t)}{\partial z^2} + \frac{\partial^2 y(z,t)}{\partial t^2} = 0$$

$$z = 0: \quad y(0,t) = \frac{\partial y(0,t)}{\partial z} = 0; \quad z = 1: \quad y(1,t) = y_1(t), \quad \frac{\partial y(1,t)}{\partial z} = \varphi(t) \tag{1.1}$$

$$n(t) = \left(1+\gamma\frac{\partial}{\partial t}\right)\frac{\partial^3 y(1,t)}{\partial z^3}, \quad b(t) = -\left(1+\gamma\frac{\partial}{\partial t}\right)\frac{\partial^2 y(1,t)}{\partial z^2}$$

$$t = 0: \quad y_1(0) = \frac{dy_1(0)}{dt} = \varphi(0) = \frac{d\varphi(0)}{dt} = y(z,0) = \frac{\partial y(z,0)}{\partial t} = 0$$

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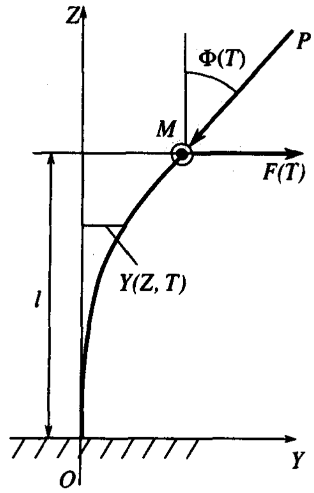

Fig. 1

Here,

$$t = T\left(\rho\frac{l^4}{EI}\right)^{-1/2}, \quad y = \frac{Y}{\delta}, \quad y_1 = \frac{Y_1}{\delta}, \quad z = \frac{Z}{l}, \quad \frac{\delta}{l} \ll 1, \quad \varphi = \frac{l}{\delta}\Phi, \quad p = \frac{l^2}{EI}P$$

$$f = \frac{l^3}{EI\delta}F, \quad m = \frac{M}{\rho l}, \quad a = \frac{A}{\rho l^3}, \quad n = \frac{l^3}{EI\delta}N, \quad b = \frac{l^2}{EI\delta}B, \quad \gamma = h\left(\rho\frac{l^4}{EI}\right)^{-1/2}$$

EI is the stiffness of the section of the rod under flexure, ρ is the linear density if the rod, h is the Voigt coefficient of internal friction, and δ is the characteristic dimension of the transverse cross-section of the rod.

2. THE DYNAMIC MODEL OF A LINEARIZED DISCRETE-CONTINUOUS SYSTEM

Suppose the functions $f(t)$, $y_1(t)$, $\varphi(t)$, $n(t)$, $b(t)$ and $y(z, t)$ satisfy the conditions for the existence of a Laplace integral transform with respect to the time t. The transforms of the equations of the linearized discrete-continuous system

$$m\lambda^2 y_1(\lambda) = n(\lambda) + f(\lambda), \quad a\lambda^2\varphi(\lambda) = b(\lambda) \tag{2.1}$$

$$\frac{\partial^4 y(z,\lambda)}{\partial z^4} + \beta(\lambda)\frac{\partial^2 y(z,\lambda)}{\partial z^2} - k^2(\lambda)y(z,\lambda) = 0$$

$$\beta(\lambda) = \frac{p}{1+\gamma\lambda}, \quad k^2(\lambda) = -\frac{\lambda^2}{1+\gamma\lambda} \tag{2.2}$$

$$z = 0: \quad y(0,\lambda) = \frac{\partial y(0,\lambda)}{\partial z} = 0; \quad z = 1: \quad y(1,\lambda) = y_1(\lambda), \quad \frac{\partial y(1,\lambda)}{\partial z} = \varphi(\lambda) \tag{2.3}$$

$$n(\lambda) = (1+\gamma\lambda)\frac{\partial^3 y(1,\lambda)}{\partial z^3}, \quad b(\lambda) = -(1+\gamma\lambda)\frac{\partial^2 y(1,\lambda)}{\partial z^2} \tag{2.4}$$

then follow from relations (1.1).

Here, $\lambda = \alpha + i\omega$, $\alpha > \alpha_0$ is the parameter of the Laplace transform and $y_1(\lambda)$, $\varphi(\lambda)$, $n(\lambda)$, $b(\lambda)$, $y(z, \lambda)$, $f(\lambda)$ are the transforms of the corresponding originals.

The general solution of the ordinary homogeneous differential equation (2.2) has the form

$$y(z,\lambda) = C_1\sin r_1 z + C_2\cos r_1 z + C_3\,\mathrm{sh}\, r_2 z + C_4\,\mathrm{ch}\, r_2 z$$
$$r_{1,2}^2(\lambda) = \pm\frac{\beta(\lambda)}{2} + \left(\left(\frac{\beta(\lambda)}{2}\right)^2 + k^2(\lambda)\right)^{1/2} \tag{2.5}$$

By satisfying the boundary conditions (2.3), we can determine the integration constants C_1, C_2, C_3 and C_4. On substituting the function $y(z, \lambda)$, which is now known, into expressions (2.4) and, then, into (2.1), we obtain a mapping of the concentrated reactions of the system, that is, of the parameters of the perturbed motion of the absolutely rigid body

$$y_1(\lambda) = W_1(\lambda)f(\lambda),\quad \varphi(\lambda) = W_2(\lambda)f(\lambda),\quad W_j(\lambda) = \frac{Q_j(\lambda)}{D(\lambda)},\quad j = 1,2$$

$$D(\lambda) = ma\lambda^4 + \left(a\frac{\xi_{11}}{\Delta} + m\frac{\xi_{12}}{\Delta}\right)\gamma\lambda^3 +$$
$$+\left(a\frac{\xi_{11}}{\Delta} + m\frac{\xi_{22}}{\Delta} + \frac{\xi_{11}\xi_{22}-\xi_{12}\xi_{21}}{\Delta^2}\gamma^2\right)\lambda^2 + \frac{\xi_{11}\xi_{22}-\xi_{12}\xi_{21}}{\Delta^2}(2\gamma\lambda+1)$$

$$Q_1(\lambda) = a\lambda^2 + \frac{\xi_{22}}{\Delta}(\gamma\lambda+1),\quad Q_2(\lambda) = -\frac{\xi_{21}}{\Delta}(\gamma\lambda+1),\quad \Delta = -\nu_{11}\nu_{22} - \nu_{12}\nu_{21}$$

$$\xi_{11} = -\nu_{22}(r_1^3\cos r_1 + r_1 r_2^2\,\mathrm{ch}\, r_2) - \nu_{21}(-r_1^3\sin r_1 + r_2^3\,\mathrm{sh}\, r_2) \tag{2.6}$$
$$\xi_{12} = \nu_{11}(-r_1^3\sin r_1 + r_2^3\,\mathrm{sh}\, r_2) - \nu_{12}(r_1^3\cos r_1 + r_1 r_2^2\,\mathrm{ch}\, r_2)$$
$$\xi_{21} = \nu_{22}(r_1^2\sin r_1 + r_1 r_2\,\mathrm{sh}\, r_2) + \nu_{21}(r_1^2\cos r_1 + r_2^2\,\mathrm{ch}\, r_2)$$
$$\xi_{22} = -\nu_{11}(r_1^2\cos r_1 + r_2^2\,\mathrm{ch}\, r_2) + \nu_{12}(r_1^2\sin r_1 + r_1 r_2\,\mathrm{sh}\, r_2)$$

$$\nu_{11} = \sin r_1 - \frac{r_1}{r_2}\mathrm{sh}\, r_2,\quad \nu_{12} = \frac{\nu_{21}}{r_1} = \cos r_1 - \mathrm{ch}\, r_2,\quad \nu_{22} = r_1\sin r_1 + r_2\,\mathrm{sh}\, r_2$$

Here, $D(\lambda)$ is a characteristic quasi-polynomial, $Q_j(\lambda)$ are the perturbing quasi-polynomials and $W_j(\lambda)$ are the concentrated transfer functions in the form of quasi-rational fractions.

Then, on introducing the transforms of the concentrated reactions $y_1(\lambda)$ and $\varphi(\lambda)$, which, according to expressions (2.6) are then known, into the relations for the constants of integration C_1, C_2, C_3, C_4 and substituting into Eq. (2.5), we obtain the transform of the distributed reaction of the system

$$y(z,\lambda) = W(z,\lambda)f(\lambda),\quad W(z,\lambda) = \frac{Q(z,\lambda)}{D(\lambda)}$$

$$Q(z,\lambda) = -\frac{a}{\Delta}[\mu_1(z,\lambda)\nu_{22} + \mu_2(z,\lambda)\nu_{21}]\lambda^2 +$$
$$+\frac{1}{\Delta^2}[\mu_1(z,\lambda)(\nu_{12}\xi_{21} - \nu_{22}\xi_{22}) - \mu_2(z,\lambda)(\nu_{11}\xi_{21} + \nu_{21}\xi_{22})](\gamma\lambda+1) \tag{2.7}$$

$$\mu_1(z,\lambda) = \sin r_1 z - \frac{r_1}{r_2}\mathrm{sh}\, r_2 z,\quad \mu_2(z,\lambda) = \cos r_1 z - \mathrm{ch}\, r_2 z$$

where $Q(z, \lambda)$ is the distributed perturbing quasi-polynomial and $W(z, \lambda)$ is the distributed transfer function.

Note that the concentrated transfer functions $W_j(\lambda)$ and the distributed transfer function $W(z, \lambda)$ are respectively the transforms of the transient impulse functions $q_j(t)$, concentrated with respect to the output $y_1(\lambda)$, $\varphi(\lambda)$ and of the transient impulse function $q(z, t)$, distributed according to the output $y(z, t)$, of the linearized discrete-continuous system perturbed by the Dirac function $f(t) = \delta(t)$. In this case, $Q(1, \lambda) = Q_1(\lambda)$, $W(1, \lambda) = W_1(\lambda)$.

Expressions (2.6) and (2.7) define a discrete-continuous system with a dynamic model of the rod where the whole infinite spectrum of characteristic frequencies and mode of vibration of the rod are taken into account in terms of the variable coefficients $\xi_{\nu j}\Delta^{-1}(\nu, j = 1, 2)$, $\mu_1(z, t)\nu_{22}\Delta^{-1}$, $\mu_2(z, t)\nu_{21}\Delta^{-1}$, $\mu_1(z, \lambda)(\nu_{12}\xi_{21} - \nu_{22}\xi_{22})\Delta^{-2}$, $\mu_2(z, \lambda)(\nu_{11}\xi_{21} - \nu_{21}\xi_{22})\Delta^{-2}$.

3. THE STABILITY AND TRANSIENT IMPULSE FUNCTIONS OF THE DYNAMIC MODEL OF A NON-CONSERVATIVE DISCRETE-CONTINUOUS SYSTEM

We will investigate the stability of the dynamic model (2.6). Note that the functions $\xi_{\nu j}(\lambda)/\Delta(\lambda)$ $(\nu, j = 1, 2)$ are analytic when $\mathrm{Re}\lambda \geq 0$, that the equalities

$$\frac{\xi_{\nu j}(-i\omega)}{\Delta(-i\omega)} = \frac{\xi_{\nu j}(i\omega)}{\Delta(i\omega)}, \quad \nu, j = 1, 2 \tag{3.1}$$

hold and that the limits

$$\lim_{\lambda\to\infty}\frac{\xi_{\nu j}}{\Delta\lambda^{\beta_{\nu j}}} = b_{\nu j}, \quad \nu, j = 1, 2 \tag{3.2}$$

$$\beta_{11} = 3/4, \quad b_{11} = -\sqrt{2}\gamma^{-3/4}, \quad \beta_{12} = 1/2, \quad b_{12} = -\gamma^{-1/2}$$

$$\beta_{21} = 1/2, \quad b_{21} = -\gamma^{-1/2}, \quad \beta_{22} = 1/4, \quad b_{22} = \sqrt{2}\gamma^{-1/4}$$

exist when $\mathrm{Re}\lambda \geq 0$.

In accordance with condition (3.1), the equalities

$$\begin{gathered}\mathrm{Re}D(-i\omega) = \mathrm{Re}D(i\omega), \quad \mathrm{Im}D(-i\omega) = -\mathrm{Im}D(i\omega)\\ \mathrm{Re}Q_j(-i\omega) = \mathrm{Re}Q_j(i\omega), \quad \mathrm{Im}Q_j(-i\omega) = -\mathrm{Im}Q_j(i\omega); \quad j = 1, 2\end{gathered} \tag{3.3}$$

hold.

According to relations (3.2), real numbers χ, β and σ exist such that

$$\begin{gathered}\text{when } \mathrm{Re}\lambda \geq 0 \quad \lim_{\lambda\to\infty}\frac{D(\lambda)}{\lambda^{n+\chi}} = c_0, \quad \lim_{\lambda\to\infty}\frac{Q_1(\lambda)}{\lambda^{k+\beta}} = c_1, \quad \lim_{\lambda\to\infty}\frac{Q_2(\lambda)}{\lambda^{s+\sigma}} = c_2\\ n+\chi > k+\beta+1, \quad n+\chi > s+\sigma+1, \quad |c_1| < \infty, \quad |c_2| < \infty, \quad c_0 \neq 0\end{gathered} \tag{3.4}$$

where n, k and s are integral powers and χ, β and σ are the increments in the degrees of the quasi-polynomials $D(\lambda)$, $Q_1(\lambda)$, $Q_2(\lambda)$ respectively when $\lambda \to \infty$, $\mathrm{Re}\lambda \geq 0$.

We now note the cases in which relations (3.4) are satisfied:

(1) if $a \neq 0$, $m \neq 0$, $\gamma \neq 0$, then

$$\begin{gathered}n = 4, \quad \chi = 0, \quad c_0 = ma, \quad k = 2, \quad \beta = 0,\\ c_1 = a, \quad s = 1, \quad \sigma = 1/2, \quad c_2 = -b_{21}\gamma\end{gathered} \tag{3.5}$$

(2) if $a = 0$, $\gamma \neq 0$, then

$$n = \begin{cases}3, & m \neq 0,\\ 2, & m = 0\end{cases}, \quad \chi = \begin{cases}1/2, & m \neq 0,\\ 1, & m = 0,\end{cases} \quad c_0 = \begin{cases}m\gamma b_{12}, & m \neq 0,\\ (b_{11}b_{22} - b_{12}b_{21})\gamma^2, & m = 0\end{cases} \tag{3.6}$$

$$k = 1, \quad \beta = 1/4, \quad c_1 = b_{22}\gamma, \quad s = 1, \quad \sigma = 1/2, \quad c_2 = -b_{21}\gamma$$

Hence, relations (3.4) and (3.3) are satisfied in the above-mentioned cases and, according to the well-known definition [4, 5], the quasi-rational fractions $W_j(\lambda)$ are physically possible. Moreover, the functions $D(\lambda)$, $Q_j(\lambda)$ are analytic on the imaginary axis and in the right half of the complex plane (λ). Consequently, according to the theorems in [4, 5] on the stability of quasi-rational fractions, the dynamic model (2.6) is asymptotically stable if the characteristic quasi-polynomial $D(\lambda)$ is stable, that is, if all of its roots lie to the left of the imaginary axis in the complex plane (λ). If just one of the roots of the quasi-polynomial $D(\lambda)$ lies to the right of the imaginary axis of the complex plane (λ), then the dynamic model (2.6) is unstable. Since the function $D(\lambda)$ is analytic on the imaginary axis and in the right half of the complex plane $\lambda = \alpha + i\omega$ and, according to relations (3.3) and (3.4), the conditions

$$\text{when } \operatorname{Re}\lambda \geq 0 \quad \lim_{\lambda\to\infty}\frac{D(\lambda)}{\lambda^{n+\chi}} = c_0 \neq 0$$

$$\forall\omega \in (-\infty, \infty): \quad D(i\omega) = u(\omega) + i\upsilon(\omega) \neq 0, \quad u(-\omega) = u(\omega), \quad \upsilon(\omega) = -\upsilon(\omega)$$

are satisfied, then, according to the theorem on the stability of a quasi-polynomial [4], all of the roots of the quasi-polynomial $D(\lambda)$ will be located to the left of the imaginary axis of the complex plane (λ) if, as ω increases monotonically from 0 to ∞, the vector $D(i\omega)$ turns in the $(u, i\upsilon)$-plane from the positive real semi-axis in a positive direction through an angle of $(n + \chi)\pi/2$, that is, an increment of argument

$$\phi = \underset{0\leq\omega\leq\infty}{\Delta} \arg D(i\omega) = (n+\chi)\pi/2 \tag{3.7}$$

is obtained.

It follows from the proof of the above-mentioned theorem presented in [4] that, in the case of an unstable quasi-polynomial $D(\lambda)$ when N of its roots are located in the right half-plane of (λ), the vector $D(i\omega)$ obtains an increment of argument

$$\phi = \underset{0\leq\omega\leq\infty}{\Delta} \arg D(i\omega) = (n+\chi)\pi/2 - N\pi \tag{3.8}$$

We now return to the distributed dynamic model (2.7). It can be seen that the equalities

$$y(\lambda) = W(z,\lambda)f(\lambda), \quad W(z,\lambda) = Q(z,\lambda)/D(\lambda)$$

hold at any fixed point $z \in (0, 1]$ of the median line of the rod.

All the arguments relating to the stability of the quasi-rational fractions $W_j(\lambda)$ will also hold in the case of the quasi-rational fractions $W(z, \lambda)$.

Consequently, the dynamic model of a linearized, non-conservative discrete-continuous system being considered is asymptotically stable in cases (3.5) and (3.6) if the equality (3.7) is satisfied and all of the roots of the quasi-polynomial $D(\lambda)$ lie in the plane (λ) to the left of the imaginary axis. Furthermore, the location of the hodograph of the vector $D(i\omega) = u(\omega) + i\upsilon(\omega)$ in the $(u, i\upsilon)$-plane when $0 \leq \omega < \infty$ as a function of the increasing follower force p enables one to make a judgment concerning the boundary of the asymptotic stability domain to which the critical value of the follower force $p = p_*$ corresponds. When $p > p_*$, the system is unstable, the roots of the characteristic quasi-polynomial $D(\lambda)$ transfer into the right half of the complex plane (λ) and the number N of these roots can be determined using relation (3.8).

Note that condition (3.4) is not satisfied in the case when $\gamma = 0$ and the quasi-rational fractions $W_j(\lambda)$ are not physically possible. This is in accordance with the known conclusion [2] concerning the fact that a model of a non-conservative system is inadmissible when $\gamma = 0$ and a quasi-critical force, which differs from the true critical force p_*, calculated when $\gamma \neq 0$, corresponds to it. The results of the analysis of the dynamic model (2.6) using the transfer function $W_1(\lambda) = Q_1(\lambda)/D(\lambda)$ in cases (3.5) and (3.6) when $\gamma \neq 0$ are presented below.

The frequency hodographs of the vector $D(i\omega)$, $0 \leq \omega < \infty$ in the $(u, i\upsilon)$-plane are shown in Fig. 2 as a function of the magnitude of the follower force p for the case of a viscoelastic rod with an absolutely rigid body on the end when $\gamma = 0.1$, $m = 1$, $a = 0.4$, $n = 4$, $\chi = 0$. When $p = 3.13 < p_*$ (curve 1), according to expression (3.7), we have $\phi = 2\pi$, and the system is asymptotically stable. When $p = p_* = 4.13$ (curve 2), the line of the hodograph passes through the point (0, 0), and the system lies on the boundary of stability. When $p = 5.13 > p_*$ (curve 3), we have $\phi = 0$, that is, according to expression (3.8), two roots of the characteristic quasi-polynomial $D(\lambda)$ transferred into the right half of the complex

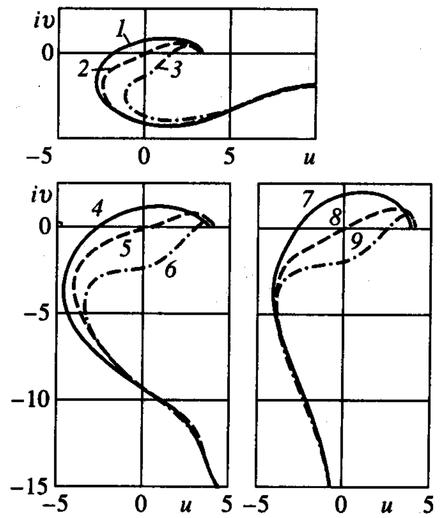

Fig. 2

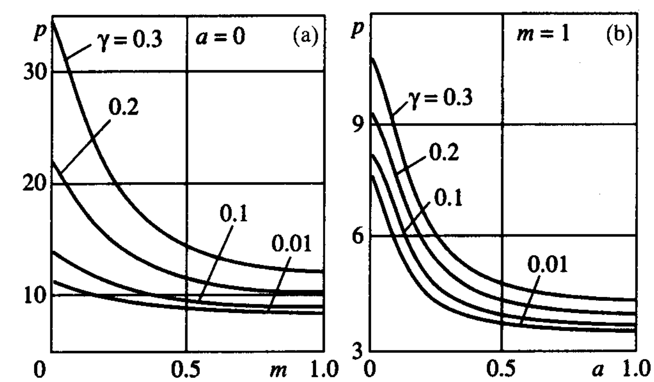

Fig. 3

plane (λ), and the system became unstable. Similarly, in the case of a viscoelastic rod with a concentrated mass on the end $\gamma = 0.1, m = 1, a = 0, n = 3, \chi = 0.5$, the hodographs of the vector $D(i\omega)$, $0 \le \omega < \infty$ are shown for an asymptotically stable system when $p = 5.263 < p_* = 8.263$, $\phi = 7\pi/4$ (curve 4), then for a system on the boundary of stability when $p = p_* = 8.263$ (curve 5) and for an unstable system when $p = 11.263 > p_* = 8.263$, $\phi = -\pi/4$ (curve 6).

As can be seen from the hodographs of $D(i\omega)$, in the case of a viscoelastic rod without a load on the end $\gamma = 0.1, m = 0, a = 0, n = 2, \chi = 1$, the rod is asymptotically stable when $p = 10.38 < p_* = 13.64$, $\phi = 3\pi/2$ (curve 7), it is on the boundary of stability $p = p_* = 13.64$ (curve 8), and the rod is unstable when $p = 15.38 > p_* = 13.64$, $\phi = \pi/2$ (curve 9). All of the hodographs of $D(i\omega)$ considered above are presented in the special scale

$$u + iv = D(i\omega)(\mathrm{Arsh}|D(i\omega)|)/|D(i\omega)|$$

The boundaries of the stability domains for different coefficients of internal friction γ for a rod with a concentrated mass m on the end when $a = 0$ in the plane of the parameters (m, p) have been constructed in Fig. 3(a). The stability domains are located below the corresponding lines. It can be seen that an increase in the mass m and a decrease in the coefficient of internal friction reduces the value of the critical follower force p_* and substantially reduces the stability domain. However, when $\gamma = 0.01$, the stability domain reaches its smallest asymptotic value and, when there is a further decrease in γ to the very small value of $\gamma = 0.0001$, the line of the boundary of the stability domain stays practically unchanged. Note that, when $m = 0$ and $\gamma = 0.0001$, the critical force has the value $p_* = 10.96$, that is,

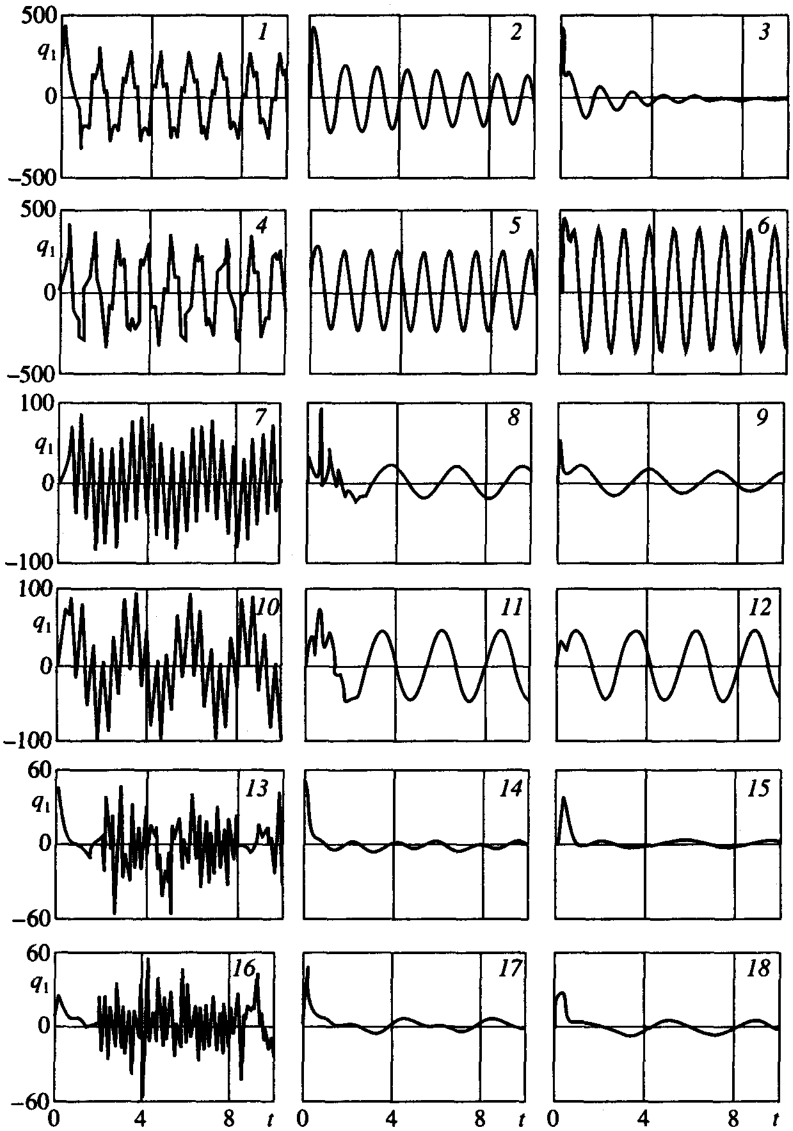

Fig. 4

it exceeds the critical force $p_* = 9.328$ [3], calculated using an approximate model with an approximation employing the first two natural modes, by 17.5%.

The boundaries of the stability domains for different values of the coefficient γ are shown in the plane of the parameters (a, p) in Fig. 3(b) for a rod with an absolutely rigid body of mass $m = 1$ and moment of inertia $a \in [0, 1]$ fixed on the end. It can be seen that a decrease in γ and an increase in a substantially reduces the value of the critical follower force. For example, $p_* = 3.7$ when $m = 1$, $a = 1$, $\gamma = 0.01$.

Suppose $f(t) = \delta(t)$ is the Dirac delta function. Then, the reaction of the discrete-continuous system at the output $y_1(t)$ to a given perturbation is the concentrated transient impulse function, which was previously denoted by $q_1(t)$. Since the transfer function $W_1(\lambda)$ is the transform off the concentrated transient impulse function $q_1(t)$ (the Laplace integral) with an abscissa of absolute convergence $\alpha = \sigma$, then, using a Mellin integral, we have

$$q_1(t) = \frac{1}{2\pi i}\int_{\alpha_0 - i\infty}^{\alpha_0 + i\infty} W(\lambda)e^{\lambda t}d\lambda, \quad \alpha_0 > \sigma, \quad t \geq 0 \tag{3.9}$$

The concentrated transient impulse functions $q_1(t)$ were calculated using an efficient algorithm [6] for values of the coefficient of internal friction $\gamma = 0.0001$, 0.01 and 0.1 in the rod for the cases of an asymptotically stable system $p < p_*$ and a system on the boundary of stability $p = p_*$ for different types of loading and, also, without a load on the end of the rod.

The calculated transient impulse functions for the different sets of values of the parameters (the corresponding number of a set is indicated below in brackets) are shown in Fig. 4. For a rod without a load on the end ($m = a = 0$) when $p = 8$, the transient impulse functions are asymptotically stable for the cases when $\gamma = 0.0001$ (1), $\gamma = 0.01$ (2), $\gamma = 0.1$ (3) (the first three graphs). An increase in the coefficient γ leads to a smoothing out of the high frequency modes and to a decrease in the amplitude of the vibrations and the time of the transient. When $p = p_* = 10.96$, $\gamma = 0.0001$ (4); $p = p_* = 10.96$, $\gamma = 0.01$ (5); $p = p_* = 13.63$, $\gamma = 0.1$ (6) (the following three graphs), the transient impulse functions at the boundary of stability take the form of non-decaying vibrations. When $\gamma = 0.01$, the amplitude of the high frequency modes is negligibly small and the end of the rod vibrates in the fundamental mode with an amplitude of 300 and a frequency of 0.85. When $\gamma = 0.0001$, the amplitude of the high frequency modes reaches a value of 100 and, consequently, the amplitude of the non-decaying transient impulse function increases up to 400.

The asymptotically stable transient impulse functions are shown for a rod with a concentrated mass $m = 1$, $a = 0$ on the end when $p = 6 \leq p_*$ for the cases $\gamma = 0.0001$ (7), $\gamma = 0.01$ (8), $\gamma = 0.1$ (9) and, also, the transient impulse functions on the boundary of stability for the cases $p = p_* = 7.9$, $\gamma = 0.0001$ (10); $p = p_* = 7.9$, $\gamma = 0.01$ (11); $p = p_* = 8.26$, $\gamma = 0.1$ (12). Similarly, the asymptotically stable transient impulse functions for a rod with an absolutely rigid body $m = 1$, $a = 0.4$ on the end when $p = 3 < p_*$ are shown for the cases $\gamma = 0.0001$ (13), $\gamma = 0.01$ (14), $\gamma = 0.1$ (15) and, also, the transient impulse functions for the cases $p = p_* = 4$, $\gamma = 0.0001$ (16); $p = p_* = 4$, $\gamma = 0.01$ (17); $p = p_* = 4.13$, $\gamma = 0.1$ (18) on the boundary of stability.

It is clear from the graphs in Fig. 4 that an increase in the coefficient of internal friction γ smooths out the high-frequency modes of vibration and that an increase in the follower force p leads to some increase in the frequency and amplitude of the fundamental (lowest) mode of vibration.

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